Early cosmology and the stochastic gravitational wave background

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The epoch when the Universe had a temperature higher than a GeV is long before any time at which we have reliable observations constraining the cosmological evolution. For example, the occurrence of a second burst of inflation (sometimes called thermal inflation) at a lower energy scale than standard inflation, or a short epoch of early matter domination, cannot be ruled out by present cosmological data. The cosmological stochastic gravitational wave background, on scales accessible to interferometer detection, is sensitive to non-standard cosmologies of this type. We consider the implications of such alternative models both for ground-based experiments such as LIGO and space-based proposals such as LISA. We show that a second burst of inflation leads to a scale-dependent reduction in the spectrum. Applied to conventional inflation, this further reduces an already disappointingly low signal. In the pre big bang scenario, where a much more potent signal is possible, the amplitude is reduced but the background remains observable by LISA in certain parameter space regions. In each case, a second epoch of inflation induces oscillatory features into the spectrum in a manner analogous to the acoustic peaks in the density perturbation spectrum. On LIGO scales, perturbations can only survive through thermal inflation with detectable amplitudes if their amplitudes were at one time so large that linear perturbation theory is inadequate. Although for an epoch of early matter domination the reduction in the expected signal is not as large as the one caused by a second burst of inflation, the detection in the context of the pre big bang scenario may not be possible since the spectrum peaks around the LIGO frequency window and for lower frequencies behaves as f^3.

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I. INTRODUCTION

A stochastic gravitational wave background is potentially a powerful source of cosmological information [1], since gravitational waves decoupled from the matter in the Universe at an extremely early stage. Various ways exist in which such a background might be generated, including cosmological inflation, collision of bubbles during a phase transition, and motions of topological defects. We will be concentrating on the first of these.

There are currently four large-scale ground-based interferometers under construction, namely LIGO, VIRGO, GEO and TAMA, and a proposal for a space-based detector LISA is also under consideration; see Ref. [1] for reviews of these experiments. Unlike measures of density perturbations, these experiments probe frequencies which are high enough that the gravitational waves have had a wavelength less than the Hubble radius since very early in the history of the Universe. Assuming the standard cosmology, where the Universe was radiation dominated from an early stage, giving way to matter domination relatively recently, the time when a comoving frequency f_* equalled the Hubble scale is given by

$$\frac{f_*}{f_0} = \frac{H_* a_*}{H_0 a_0} \approx \frac{T_*}{T_{\rm eq}} \, z_{\rm eq}^{1/2} \,, \tag{1}$$

where $f_0 = a_0 H_0 = 3h \times 10^{-18}$ Hz is the mode that is just re-entering the Hubble radius today, T is the temperature and $z_{\rm eq} = 24\,000\,\Omega_0 h^2$ the redshift of matter–radiation equality. Here Ω_0 and h are the density parameter and Hubble constant in the usual units. Comoving units have been normalized to match present physical units. Since $T_{\rm eq} = 24\,000\,\Omega_0 h^2\, T_0 \approx 1{\rm eV}$, we have

$$\frac{f_*}{100\,{\rm Hz}} \approx \frac{T_*}{10^9\,{\rm GeV}} \,, \tag{2}$$

where $100\,{\rm Hz}$ is the typical frequency band probed by ground-based interferometers.

The epoch when the Universe had a temperature higher than a GeV is long before any time at which we have reliable observations constraining the cosmological evolution. Because interesting wavelengths have been less than the Hubble

radius since then, the amplitude of the stochastic background will be sensitive to whatever the actual cosmological evolution is. A different cosmological model will have two principal effects: first, it will modify the relationship between scales given by Eq. (2) for the standard cosmology, and secondly it can alter the amplitude. Only those scales which are larger than the Hubble radius will have their amplitude unchanged in alternative cosmologies.

To highlight this effect, we shall consider two alternative scenarios to the standard cosmology. In the first we assume a second short burst of inflation at a lower energy scale than standard inflation, often called thermal inflation [2]. Such a short stage of inflation may be desirable to get rid of unwanted relic particles which could otherwise have dramatic consequences on the subsequent evolution of the Universe [2]. This second burst of inflation has in general no connection to the usual inflationary epoch which is assumed responsible for the generation of the density perturbations and gravitational waves leading to structure formation and microwave anisotropies. It is normally imagined to begin at a temperature around 10^7 GeV, and continue until about 10^3 GeV (the supersymmetry scale), giving a total of $\ln 10^4 \simeq 10$ e-foldings of inflation. If this second burst of inflation occurs after the scales of interest (i.e. the scales accessible to LIGO and LISA) are within the horizon, then the stochastic background is altered.

The second alternative is the recently discussed possibility that the Universe becomes temporarily dominated by some long-lived massive particle (for example the moduli fields of string theory), inducing an early period of matter domination which ends when the particle finally decays to restore the usual radiation-dominated Universe (see Ref. [3] and references therein). Another possible origin for a short period of early matter domination lies with the QCD phase transition, as was recently discussed by Schwarz [4].

To make the discussion concrete, we require a model for the initial gravitational wave spectrum. We consider two somewhat different models for the actual origin of the stochastic background. The first is conventional inflation, which is the standard case but which gives a disappointingly low signal even in the standard cosmology. The second is an unusual variant on inflation known as the pre big bang (PBB) scenario [5,6]. This is a much more speculative idea, but potentially of greater interest as it predicts a spectrum which rises sharply towards small scales [7,8], the opposite of the usual inflationary behaviour, thus creating a realistic prospect of direct detection of the gravitational wave background [9].

II. COSMOLOGY WITH THERMAL INFLATION

The first modification to the standard Big Bang model we consider is a second epoch of inflation occurring after the beginning of the radiation epoch. We assume that the original epoch of inflation (either conventional or pre big bang) responsible for generating the gravitational wave background ends at conformal time τ_{r1} (from now on we will always use conformal time), at which point the radiation era begins. At time τ_{i2} a second era of inflation starts which ends at τ_{r2}; we will model this era as exponential inflation. After τ_{r2} the Universe again becomes radiation dominated and the evolution of the Universe continues according the standard model with an era of matter domination starting at τ_m which lasts up to the present. The scale factor after the original period of inflation is given by

$$
a(\tau) = \begin{cases}
a_2 \left(\tau_2 + \tau \right); & \tau_{r1} < \tau < \tau_{i2} \\[2ex]
a_3 \left(\tau_3 - \tau \right)^{-1}; & \tau_{i2} < \tau < \tau_{r2} \\[2ex]
a_4 \tau; & \tau_{r2} < \tau < \tau_{eq} \\[2ex]
a_5 \left(\tau_5 + \tau \right)^2; & \tau > \tau_{eq}
\end{cases}
\tag{3}
$$

where the constants are determined by requiring the continuity of a and $da/d\tau$ at the transition between two epochs. For the moment we only want to make an approximate estimate of the modifications introduced in the spectrum of the stochastic gravitational wave background and we will not worry about the exact value of these constants. We parametrize the thermal inflation by the energy scale at which it starts, ρ_{i2}, and by the total expansion it gives rise to, $R_i = a_{r2}/a_{i2}$.

In this model we assume that the first epoch of radiation domination happens at a temperature of order 10^{16} GeV, while the short period of thermal inflation begins around 10^8 GeV. We also assume that during thermal inflation the Universe expands by a factor of 10^4 (approximately 10 e-foldings of inflation). This is the minimal amount of inflation which solves the moduli problem [2].

Normalizing the scale factor at the present as $a_0 = 1$, we can write all the relevant parameters in our model in terms of H_i, H_{i2} (the Hubble parameters at the end of the first period of inflation and at beginning of the second stage of inflation), R_i, the present Hubble parameter H_0 and the redshift at the matter–radiation equality z_{eq}. In terms of these parameters we can write the Hubble parameter and time at the matter–radiation equality

$$H_{\text{eq}} = (1 + z_{\text{eq}})^{3/2} H_0 \,, \quad \tau_{\text{eq}} = (1 + z_{\text{eq}})^{-1/2} H_0^{-1} \,, \tag{4}$$

the Hubble parameter, redshift and time at the end of the second epoch of inflation

$$H_{\text{r2}} = H_{\text{i2}} \,, \quad 1 + z_{\text{r2}} = (1 + z_{\text{eq}})^{1/4} \left(\frac{H_{\text{i2}}}{H_0} \right)^{1/2} \,, \quad \tau_{\text{r2}} = \frac{(1 + z_{\text{eq}})^{1/4}}{(H_{\text{i2}} H_0)^{1/2}} \,, \tag{5}$$

as well as the redshift and time at the beginning of the second period of inflation

$$1 + z_{\text{i2}} = (1 + z_{\text{eq}})^{1/4} R_{\text{i}} \left(\frac{H_{\text{i2}}}{H_0} \right)^{1/2} \,, \quad \tau_{\text{i2}} \approx -\frac{R_{\text{i}} (1 + z_{\text{eq}})^{1/4}}{(H_{\text{i2}} H_0)^{1/2}} \,, \tag{6}$$

and at the end of the first epoch of inflation

$$1 + z_{\text{r1}} = (1 + z_{\text{eq}})^{1/4} R_{\text{i}} \left(\frac{H_{\text{r1}}}{H_0} \right)^{1/2} \,, \quad \tau_{\text{r1}} \approx -\frac{2 R_{\text{i}} (1 + z_{\text{eq}})^{1/4}}{(H_{\text{i2}} H_0)^{1/2}} \,. \tag{7}$$

We now describe the physical effects through which thermal inflation modifies the spectrum. The technical details of the calculation are quite complex, and we describe them fully in the Appendix; here in the main text we will restrict ourselves to describing the relevant physics and exhibiting the results.

On short and long scales things are relatively simple. Modes always above the horizon cannot change their amplitude, but there is an apparent effect caused by the change in the correspondence between comoving scales (i.e. physical scales during inflation) and present physical scales; the extra expansion during thermal inflation shifts the spectrum to long wavelengths. For a flat spectrum this has no effect, but if the spectrum is tilted this alters the amplitude on a given present physical scale. This effect may have observational consequences in the contribution of tensor perturbations to the anisotropy of the cosmic microwave background radiation as we will discuss in the following subsection.

A related effect is a redshift in the high-frequency cutoff of the spectrum corresponding to the horizon size at the end of the main period of inflation. From Eq. (A21) we obtain

$$f_{\text{r1}} = \frac{(H_{\text{r1}} H_0)^{1/2}}{(1 + z_{\text{eq}})^{1/4} R_{\text{i}}} \,. \tag{8}$$

Very short scale modes, which are always inside the horizon when the alternate behaviour is occurring, simply suffer an additional redshift. Modes inside the horizon behave as radiation, with the energy density decreasing as a^{-4}. If thermal inflation occurs it means there is more total expansion between the initial generation of the modes and the present, and so their amplitude is suppressed. For example, with perfect exponential thermal inflation, this suppression will be $R_{\text{i}}^{-4} \sim 10^{-16}$. Had we assumed an epoch of power-law thermal inflation with $a \propto \tau^{-q}$ ($q \geq 1$), f_{r1} would be redshifted by $R_{\text{i}}^{(1+q)/2q}$. Recalling that during an epoch of thermal inflation the total energy density redshifts as $a^{2(1/p-1)}$, the suppression in Ω_{gw} in this case would be by a factor $\rho_{\text{gw}}/\rho = R_{\text{i}}^{-2(1+1/q)}$.

The most interesting regime is the intermediate one. For waves with frequencies in the interval $f_{\text{i2}} < f < f_{\text{r2}}$, a curious phenomenon happens: these waves enter the Hubble radius during the first epoch of radiation but are again pushed outside the horizon during the subsequent epoch of inflation. While they are inside the horizon their behaviour is oscillatory. Hence, when they leave the horizon, their phases will have a strong dependence on frequency and this will leave an oscillatory imprint in the spectrum. This is strongly reminiscent of the acoustic peak structure in the density perturbation spectrum which leads to oscillations in the radiation power spectrum, and is arising similarly from the dominance of growing mode perturbations at horizon entry. Similar oscillations have also been found in the primordial density perturbation spectrum in the case of double inflation [10].

To make our discussion concrete, we now exhibit results for two particular models for the initial spectrum.

A. Conventional inflation

We model conventional inflation as a period of power-law inflation [11], corresponding to an initial epoch where the scale factor behaves as

$$a(\tau) = a_1 (\tau_1 - \tau)^{-p} \,; \quad -\infty < \tau < \tau_{\text{r1}} \,. \tag{9}$$

where $p \geq 1$ (equality holds for exponential inflation). When we write the scale factor in comoving time we get $a \propto t^{\alpha}$ with p and α related by $p = \alpha/(\alpha - 1)$.

In this standard inflationary scenario the spectral energy density parameter for the gravitational waves can be calculated in the long-wavelength approximation where we assume that $k\tau \ll 1$ (see the Appendix for the technical details). After a straightforward but tedious calculation we obtain

$$\Omega_{\text{gw}} = \begin{cases} \dfrac{2^{3p-2}}{3^p} \dfrac{\pi^{1-p}\,\Gamma}{p^{2(1+p)}} \dfrac{(\rho_{\text{c}}\,\rho_{\text{r}})^{(1+p)/2}}{\rho_{\text{Pl}}^p} \left(\dfrac{f_{\text{Pl}}}{f}\right)^{2p} \quad ; & H_0 \ < f < \ H_0(1+z_{\text{eq}})^{1/2} \\[18pt] \dfrac{2^{3p+1}}{3^{p+1}} \dfrac{\pi^{2(1-p/2)}\,\Gamma}{p^{2(1+p)}} \dfrac{\left(\rho_{\text{c}}^{p-1}\rho_{\text{r}}^{p+1}\right)^{1/2}}{\rho_{\text{Pl}}^p\,(1+z_{\text{eq}})^{(1+p)/2}} \left(\dfrac{f_{\text{Pl}}}{f}\right)^{2(p-1)}; & H_0(1+z_{\text{eq}})^{1/2} \ < f < \ \dfrac{(H_{\text{i}}H_0)^{1/2}}{(1+z_{\text{eq}})^{1/4}}\, . \end{cases} \tag{10}$$

where Γ is given by

$$\Gamma = \frac{\left[\Gamma\!\left(-\frac{1}{2}-p\right) + 2\Gamma\!\left(\frac{1}{2}-p\right)\right]^2}{\left[\Gamma\!\left(-\frac{1}{2}-p\right)\Gamma\!\left(\frac{1}{2}-p\right)\right]^2} \sec^2(p\pi)$$

When $p = 1$ this gives $\Gamma = 1/\pi$ and we recover the usual result for exponential inflation.

From the observational point of view, the extra redshift suffered by waves inside the horizon during the second stage of inflation will have a devastating effect on the prospects for observing them. Assuming for instance that during the second epoch of inflation we have exponential inflation, we will have an extra redshift of R_{i}^4, which assuming $R_{\text{i}} \approx 10^4$ gives a redshift of 10^{16}. Since LIGO operates in the KHz region, which corresponds to energies of the order of 10^7 GeV, only when the second burst of inflation occurs at an energy scale larger than this can we ensure that the LIGO region will not be affected. However, in the current scenario, the second epoch of inflation will be useful only when it occurs at energy scales lower than this. As for LISA, depending on the scale at which the second epoch of inflation occurs the situation may not worsen, because LISA scales may remain superhorizon for most or all of the thermal inflation epoch. Nevertheless the direct detection of the gravitational wave background by LISA was already impossible in conventional inflationary models, because the predicted signal is too low even before thermal inflation does further damage.

The exact result for the spectral energy density parameter when both the initial and second epoch of inflation (thermal inflation) are exponential, obtained as described in the Appendix, is plotted in Fig. 1. This plot illustrates all the features discussed so far and also shows that the extra redshift introduced by thermal inflation makes the direct detection of the stochastic background almost impossible. In the long-wavelength approximation we get:

$$\Omega_{\text{gw}} = \begin{cases} \dfrac{2^{3p-2}}{3^p} \dfrac{\pi^{1-p}\,\Gamma}{p^{2(1+p)}} \left(\dfrac{(\rho_{\text{c}}\,\rho_{\text{r1}})^{p+1}}{\rho_{\text{Pl}}^{2(p+1)} R_{\text{i}}^{4(p-1)} (1+z_{\text{eq}})^{p-1}}\right)^{1/2} \left(\dfrac{f_{\text{Pl}}}{f}\right)^{2p} \quad ; & H_0 \ < f < \ H_0(1+z_{\text{eq}})^{1/2} \\[20pt] \dfrac{2^{3p+1}}{3^{p+1}} \dfrac{\pi^{2-p}\,\Gamma}{p^{2(1+p)}} \left(\dfrac{\rho_{\text{c}}^{p-1}\rho_{\text{r1}}^{p+1}}{\rho_{\text{Pl}}^{2p} R_{\text{i}}^{4(p-1)} (1+z_{\text{eq}})^{p+1}}\right)^{1/2} \left(\dfrac{f_{\text{Pl}}}{f}\right)^{2(p-1)}; & H_0(1+z_{\text{eq}})^{1/2} \ < f < \ \dfrac{(H_{\text{i2}}H_0)^{1/2}}{R_{\text{i}}(1+z_{\text{eq}})^{1/4}} \\[20pt] \dfrac{2^{3(p+1)}}{3^{p+3}} \dfrac{\Gamma}{p^{2(1+p)}\pi^p} \left(\dfrac{(\rho_{\text{c}}\,\rho_{\text{r1}})^{(p+1)/2}\,\rho_{\text{i2}}}{\rho_{\text{Pl}}^{p+2} R_{\text{i}}^{2(p+1)} (1+z_{\text{eq}})^{(p+3)/2}}\right) \times \\[14pt] \qquad\qquad \cos^2(2\pi f \tau_{\text{r1}}) \left(\dfrac{f_{\text{Pl}}}{f}\right)^{2(p+1)}; & \dfrac{(H_{\text{i2}}H_0)^{1/2}}{R_{\text{i}}(1+z_{\text{eq}})^{1/4}} \ < f < \ \dfrac{(H_{\text{i2}}H_0)^{1/2}}{(1+z_{\text{eq}})^{1/4}} \\[20pt] \dfrac{2^{3p+1}}{3^{p+1}} \dfrac{\pi^{2-p}\,\Gamma}{p^{2(1+p)}} \left(\dfrac{\rho_{\text{c}}^{p-1}\rho_{\text{r1}}^{p+1}}{\rho_{\text{Pl}}^{2p} (1+z_{\text{eq}})^{1+p} R_{\text{i}}^{4(p+1)}}\right)^{1/2} \left(\dfrac{f_{\text{Pl}}}{f}\right)^{2(p-1)}; & \dfrac{(H_{\text{i2}}H_0)^{1/2}}{(1+z_{\text{eq}})^{1/4}} \ < f < \ \dfrac{(H_{\text{r1}}H_0)^{1/2}}{R_{\text{i}}(1+z_{\text{eq}})^{1/4}}\, . \end{cases} \tag{11}$$

The cosine term appearing in the third branch of the spectrum deserves some further explanation. In a full long-wavelength approximation we would expand the cos function keeping only the leading order term which in this case is 1. However, in this case it can be seen that $k\tau_{\text{r1}} > 1$ and the series expansion would not be appropriate. This is the mathematical reason behind the oscillations between the two plateaus in Fig. 1.

The long-wavelength approximation in Eq. (11) fits the exact result very well and it is almost indistinguishable from it except very close to frequencies associated with a transition and during the oscillations in the central region of the spectrum. The right panel in Fig. 1 shows a comparison between the exact result and the long-wavelength approximation in the oscillating part of the spectrum. Since the frequency of the oscillations increases with $\tau_{\text{r1}} \propto R_{\text{ti}}$, in this plot we used a small value for R_{ti} in order to be able to show the oscillations in detail. In the particular
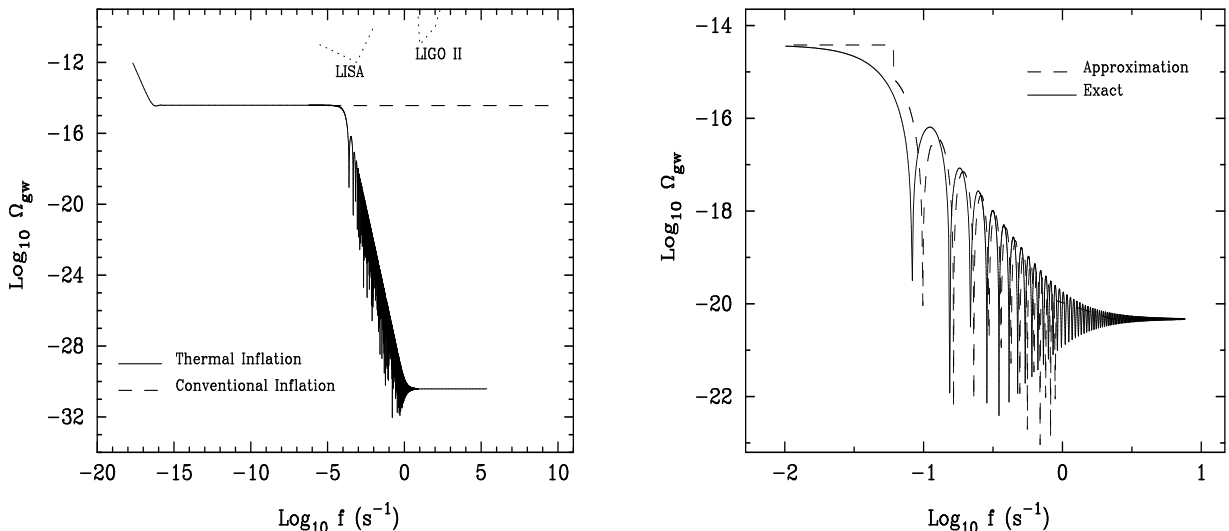
FIG. 1. The plot on the left shows the exact spectral energy density parameter for the model of thermal inflation where we have exponential inflation in both inflationary eras. The spectrum for the usual model of exponential inflation without an extra epoch of thermal inflation is also shown for comparison purposes. The dotted lines marked LISA and LIGO II are the expected sensitivities of the gravitational wave detectors LISA and LIGO in the advanced configuration. Detection was already very problematic in conventional models of inflation [12], and with the inclusion of an epoch of thermal inflation the scales relevant for LIGO are redshifted by an additional factor R_i^4 which makes their detection hopeless. This plot was obtained with $\rho_i^{1/4} = 3 \times 10^{16}$ GeV, $\rho_{i2}^{1/4} = 10^8$ GeV and $R_i = 10^4$. Here, as in all the other plots in this paper, we assume $h = 0.65$. The plot on the right shows a close-up of the oscillations in the central part of the spectrum, both for the exact result and the long-wavelength approximation given by Eq. (11). In order to be able to resolve individual oscillations, we used $R_{ti} = 30$ rather than the realistic value of 10^4 for this illustration. All the other parameters are the same as in the left plot. We checked that on increasing the number of e-foldings of expansion during thermal inflation, the phase difference between the exact result and the approximation decreases and for $R_{ti} \approx 100$ the two curves are almost indistinguishable.

case shown in the figure, the oscillations in the approximation have a different phase from those in the exact result. However, we checked that on increasing R_{ti} (and therefore the frequency of the oscillations), the phase difference decreases, until the approximation and the exact result agree very well for values of $R_{ti} \gtrsim 100$.

The factor $R_i^{4(p-1)}$ present in the two lower frequency branches of Ω_{gw} is due to the change in the correspondence between comoving and physical scales, as was discussed above. We can see that if the initial epoch is exponential inflation ($p = 1$) this effect is absent. This can be understood when we realize that the effect is a shift in the spectrum towards low frequencies. When the spectrum is flat, as is the case for exponential inflation, the net effect for frequencies which were always outside the horizon is null, while for a tilted spectrum such as the one produced by power-law inflation this will appear as a further redshift in the spectrum at these scales. This means that when the initial spectrum is generated by power-law inflation and therefore is not flat, the contribution for the anisotropies in the cosmic microwave background radiation will be suppressed. The contribution for the anisotropies coming from scalar perturbations will also be suppressed, but if the initial spectra have a different tilt there will be a net effect in the tensor to scalar ratio.

One can view the effect of thermal inflation as being a 'transfer function' which processes the initial spectrum into a final spectrum. Its rather complicated form is given by the ratio of the final Bogolubov coefficient β to the initial β at the beginning of the radiation epoch. In the long-wavelength approximation, the transfer function is independent of the initial spectrum which is being processed; for example, we expect the suppression to take exactly the same form in the pre big bang scenario, as we shortly illustrate. In the end the transfer function will lead to the same result as the full long-wavelength approximation (except possibly for a factor of $O(1)$). The advantage of the transfer function formalism is that regardless of how the initial spectrum was produced (in this subsection conventional inflation, in the next the pre big bang model) for a given evolution of the Universe after the initial spectrum was produced, the transfer function is always the same.

Let us now illustrate the use of the transfer function as discussed in the appendix. Neglecting constants of $O(1)$ the initial spectrum is in this case given, in terms of the number of particles N_k in mode k as defined in the appendix, as

$$N_k = \frac{1}{(k\tau_{pl1})^{2(p+1)}}, \tag{12}$$

5

where $\tau_{\mathrm{pl1}} = 2(\tau_{\mathrm{r2}} - \tau_{\mathrm{i2}}) + \tau_{\mathrm{r1}}$ is nothing but the argument of the Hankel functions appearing in the mode functions. From Eqs. (A27) and (A28) we easily obtain

$$
G\left(k\right) = \begin{cases} \tau_{\mathrm{i2}}^{-2}\tau_{\mathrm{r2}}^{2}\tau_{\mathrm{eq}}^{-2}k^{-2} & H_0 < k < a_{\mathrm{eq}}H_{\mathrm{eq}} \\ \tau_{\mathrm{i2}}^{-2}\tau_{\mathrm{r2}}^{2} & a_{\mathrm{eq}}H_{\mathrm{eq}} < k < a_{\mathrm{i2}}H_{\mathrm{i2}} \\ \tau_{\mathrm{r2}}^{-4}k^{-4} & a_{\mathrm{i2}}H_{\mathrm{i2}} < k < a_{\mathrm{r2}}H_{\mathrm{r2}} \\ 1 & a_{\mathrm{r2}}H_{\mathrm{r2}} < k < a_{\mathrm{r1}}H_{\mathrm{r1}} \end{cases}
\tag{13}
$$

After all the factors involving the conformal time at the different transitions are written in terms of the parameters of our model with help of Eqs. (4) to (7), we obtain the same result as in Eq. (11), except for numerical factors.

B. The pre big bang cosmology

The pre big bang (PBB) is a recently-proposed cosmological scenario [5,6] inspired by scale factor duality, one of the basic ideas of string theory [13]. We will briefly summarize here some of the main features of the PBB scenario. For a more comprehensive review, covering both the motivation and the phenomenology, see Ref. [14].

In the PBB scenario the Universe started its evolution from the most simple initial state conceivable, the perturbative vacuum of string theory. In such a state physics can be described by the tree-level low-energy effective action of string theory given by

$$
S_{\mathrm{eff}} = \frac{1}{2\lambda_{\mathrm{s}}^{2}} \int \sqrt{-g}e^{-\phi}\left[\mathcal{R} + \partial_{\mu}\phi\partial^{\mu}\phi - V(\phi)\right] d^{4}x
\tag{14}
$$

The details of the potential $V(\phi)$ are still largely unknown, except that it goes to zero as a double exponential when $\phi \to -\infty$ and it must have a non-trivial minimum at the present.

During an initial stage, the evolution of the Universe is driven by the kinetic energy of the dilaton field. At this stage both the curvature scale and the dilaton increase as $t \to 0_{-}$, $|H| \sim |\dot{\phi}| \sim t^{-1}$ and there is in this case an exact solution for both the scale factor and the dilaton [5,6,8].

$$
\begin{aligned}
a &= (-t)^{-1/2} \\
\phi &= -3\ln(-t).
\end{aligned}
\tag{15}
$$

The solution with increasing H is in fact an inflationary solution of the so-called super-inflation or pole-inflation type and this is one of the main peculiarities of this model. During this stage in the evolution of the Universe all the couplings will be very small since they are given by $g = \exp(\phi/2) \approx 0$.

As soon as the curvature reaches the string scale $M_{\mathrm{s}} \equiv \lambda_{\mathrm{s}}^{-1}$, all the higher-order derivative terms in the action become important and the effective action Eq. (14), as well as the solution Eq. (15), are no longer valid. Unfortunately during the stage where the truly stringy regime is reached not much is known. What we know however is that during the stringy era the curvature scale must grow only mildly in order to prevent its unbounded growth with the consequent singularity (the Big Bang) which we are trying to avoid in this model. Therefore, whatever the exact form of the solution is, we expect that $H \approx \mathrm{const}$ and $\dot{\phi} \approx \mathrm{const}$. The constraint on H will therefore lead us to some form of conventional inflation during the stringy regime.

At some stage during the string phase the dilaton eventually approaches the strong coupling regime $g \approx 1$ and at this point the dilaton will freeze, leaving us with the correct value of Newton's constant, and the transition to conventional cosmology starting with a radiation-dominated phase will occur.

The evolution we have described so far is based on the action Eq. (14) where the dilaton is non-minimally coupled to curvature, the so-called string frame. By applying a conformal transformation involving the dilaton we go from the string frame to the Einstein frame where we recover the usual Einstein–Hilbert action for gravity. When we transform from the string frame to the Einstein frame, the super-inflation type solution for the scale factor is transformed into accelerated contraction in the Einstein frame. Since physics is not changed by a conformal transformation, both frames are equivalent. After the string era when the dilaton freezes, the string and the Einstein frames coincide.

One of the most remarkable predictions of the PBB model is a spectrum of gravitational waves increasing with frequency [7]. The reason for this lies in the increase in the curvature during the dilaton era. Due to this characteristic shape, the cosmological background of gravitational waves produced in this type of model can in principle be detected for a broad range of parameters by the various detectors [9]. We can therefore hope that even when an epoch of thermal inflation is included in this model, the direct detection of the gravitational wave background may still be possible for certain ranges of parameters.

Without loss of generality, we will perform the entire computation of the gravitational wave background in the string frame, since the perturbation spectrum is frame-independent [15].

First of all, we must have an explicit form for the scale factor. During the string phase there is a well-known solution given in comoving time by Eq. (15), which in conformal time τ takes the form $a \propto \tau^{-1/(1+\sqrt{3})}$ and $\phi \propto -\sqrt{3}\ln\tau$. During the string era, however, all we know is that the curvature scale must be approximately constant. We can therefore model the stringy phase by an epoch of power-law inflation. The scale factor before the first radiation dominated era takes the form

$$
a(\tau) = \begin{cases} a_\mathrm{s}\left(\dfrac{\alpha\,a_\mathrm{s}}{a'_\mathrm{s}}\right)^{\alpha}(\tau_\mathrm{dil} - \tau)^{-\alpha}\,; & -\infty < \tau < \tau_\mathrm{s} \\[4mm] a_\mathrm{r1}\left(\dfrac{p\,a_\mathrm{r1}}{a'_\mathrm{r1}}\right)^{p}(\tau_\mathrm{st} - \tau)^{-p}\,; & \tau_\mathrm{s} < \tau < \tau_\mathrm{r1} \end{cases}
\tag{16}
$$

where the subscript 's' denotes the beginning of the string era, $\alpha = 1/(1+\sqrt{3})$, p is a free parameter and

$$
\begin{aligned}
\tau_\mathrm{dil} &= \tau_\mathrm{s} + \alpha\frac{a_\mathrm{s}}{a'_\mathrm{s}} \\[2mm]
\tau_\mathrm{st} &= \tau_\mathrm{r1} + p\frac{a_\mathrm{r1}}{a'_\mathrm{r1}}\,.
\end{aligned}
\tag{17}
$$

For the dilaton we have

$$
\phi(\tau) = \begin{cases} \phi_\mathrm{dil} - \sqrt{3}\ln\left(\tau_\mathrm{dil} - \tau\right)\,; & -\infty < \tau < \tau_\mathrm{s} \\[3mm] \phi_\mathrm{st} - 2\beta\ln\left(\tau_\mathrm{st} - \tau\right)\,; & \tau_\mathrm{s} < \tau < \tau_\mathrm{r1} \\[3mm] \phi_0\,; & \tau > \tau_\mathrm{r1} \end{cases}
\tag{18}
$$

The constants ϕ_dil and ϕ_st are determined by the continuity of the dilaton, while ϕ_0 is determined by the present value of G. We will not write them here since they will not be needed in what follows. As to β, it is determined by the continuity of ϕ' at the transition from the dilaton to the stringy phase and its value is given by

$$
\beta = \frac{\sqrt{3}}{2}\frac{p}{\alpha} = \frac{3+\sqrt{3}}{2}\,p\,.
\tag{19}
$$

This parameter will be important in the determination of the order μ of the Hankel functions appearing in the mode functions for the gravitons during the stringy era (see Appendix) which in this case is given by

$$
\mu = \frac{1}{2}\left|2p - 2\beta + 1\right| = \frac{p\left(1+\sqrt{3}\right) - 1}{2}
\tag{20}
$$

After the end of the string phase, a radiation era begins and from then on the evolution of the Universe is given by Eq. (3).

In this model we will have one parameter more than in the context of conventional cosmology: the total expansion during the string phase, $R_\mathrm{s} = a_\mathrm{r1}/a_\mathrm{s}$.

In this case it is not possible to make ϕ' continuous at the transition from the string phase to the radiation phase. This is the reason why we present the Bogolubov coefficients formalism in the Appendix in a form which does not require the continuity of the first derivative of R, which in this case is given by $R = ae^{-\phi/2}$.

From now on everything follows as before, except that we must use R instead of a, and the expressions for the Bogolubov coefficients have an extra term due to the discontinuity in ϕ'.

Before we proceed, let us just recall that in the absence of thermal inflation the spectrum for the gravitational waves is given by

$$
\Omega_\mathrm{gw} \approx \begin{cases} \Upsilon\left(\dfrac{8\pi}{3}\right)^{1/2}\dfrac{R_\mathrm{s}^{\delta}}{(1+z_\mathrm{eq})^{1/4}}\left(\dfrac{\rho_\mathrm{r}^{1/2}\rho_\mathrm{Pl}}{\rho_\mathrm{c}^{3/2}}\right)^{1/2}\left(\dfrac{f}{f_\mathrm{Pl}}\right)^{3} & H_0\left(1+z_\mathrm{eq}\right)^{1/2} < f < \dfrac{(H_\mathrm{r}H_0)^{1/2}}{R_\mathrm{s}^{1/q}\left(1+z_\mathrm{eq}\right)^{1/4}} \\[6mm] \Theta\left(\dfrac{2^{6\mu-1}\pi^{5-2\mu}}{3^{2\mu+1}q^{2(2\mu+1)}}\right)^{1/2}\left(\dfrac{\rho_\mathrm{c}^{2\mu-3}\rho_\mathrm{r}^{2\mu+1}}{\rho_\mathrm{Pl}^{2(2\mu-1)}\left(1+z_\mathrm{eq}\right)^{2\mu+1}}\right)^{1/4}\left(\dfrac{f_\mathrm{Pl}}{f}\right)^{2\mu-3} & \dfrac{(H_\mathrm{r}H_0)^{1/2}}{R_\mathrm{s}^{1/q}\left(1+z_\mathrm{eq}\right)^{1/4}} < f < \dfrac{(H_\mathrm{r}H_0)^{1/2}}{\left(1+z_\mathrm{eq}\right)^{1/4}} \end{cases}
\tag{21}
$$

where

$$\Upsilon = \frac{\left(1+\sqrt{3}\right)\left[p\left(2+\sqrt{3}\right)-1\right]^2\left[\pi^2+(2+2\gamma-\ln 4)^2\right]}{\left[p\left(1+\sqrt{3}\right)-1\right]^2}$$

($\gamma \simeq 0.577$ is the Euler constant) and

$$\Theta = \frac{\left[2\,\Gamma(1-\mu)-\left(\left(3+\sqrt{3}\right)q-1\right)\Gamma(\mu)\right]^2}{\Gamma^2(1-\mu)\,\Gamma^2(-\mu)}\csc^2(\mu\pi)$$

Since $\Omega_{\rm gw}$ for modes entering the horizon after the matter–radiation equality will be far too small to have any observational consequences, we will forget about this part of the spectrum. The initial spectrum, i.e. the spectrum at the beginning of the first radiation era is

$$N_{\rm k}^{(0)} = \frac{\tau_{\rm s}^{2\mu}}{\tau_{\rm r}^{2\mu+1}k} \tag{22}$$

Applying the transfer function in Eq. (13) to this equation[1] and writing all the factors of conformal time in terms of redshifts and energy scales we obtain:

$$\Omega_{\rm gw} \approx \begin{cases} \dfrac{\rho_{\rm r1}^{1/4}\rho_{\rm Pl}^{1/2}}{\rho_{\rm c}^{3/4}}\dfrac{R_{\rm s}^{\delta}R_{\rm ti}}{(1+z_{\rm eq})^{1/4}}\left(\dfrac{f}{f_{\rm Pl}}\right)^3; & H_0(1+z_{\rm eq})^{1/2} \ < f < \ \dfrac{(H_{\rm ti}H_0)^{1/2}}{R_{\rm ti}(1+z_{\rm eq})^{1/4}} \\[2ex] \dfrac{\rho_{\rm r1}^{1/4}\rho_{\rm c}^{1/4}\rho_{\rm ti}}{\rho_{\rm Pl}^{3/2}}\dfrac{R_{\rm s}^{\delta}}{R_{\rm ti}(1+z_{\rm eq})^{5/4}}\cos^2(2\pi f\tau_{r1})\left(\dfrac{f_{\rm Pl}}{f}\right); & \dfrac{(H_{\rm ti}H_0)^{1/2}}{R_{\rm ti}(1+z_{\rm eq})^{1/4}} \ < f < \ \dfrac{(H_{\rm ti}H_0)^{1/2}}{(1+z_{\rm eq})^{1/4}} \\[2ex] \dfrac{\rho_{\rm r1}^{1/4}\rho_{\rm Pl}^{1/2}}{\rho_{\rm c}^{3/4}}\dfrac{R_{\rm s}^{\delta}}{R_{\rm ti}(1+z_{\rm eq})^{1/4}}\left(\dfrac{f}{f_{\rm Pl}}\right)^3; & \dfrac{(H_{\rm ti}H_0)^{1/2}}{(1+z_{\rm eq})^{1/4}} \ < f < \ \dfrac{(H_{\rm r1}H_0)^{1/2}}{R_{\rm s}^{1/p}R_{\rm ti}(1+z_{\rm eq})^{1/4}} \\[2ex] \dfrac{\rho_{\rm r1}^{(\mu+1/2)/2}\rho_{\rm c}^{(\mu-3/2)/2}}{\rho_{\rm Pl}^{\mu-1/2}R_{\rm ti}^{2\mu+1}(1+z_{\rm eq})^{(\mu+1/2)/2}}\left(\dfrac{f_{\rm Pl}}{f}\right)^{2\mu-3}; & \dfrac{(H_{\rm r1}H_0)^{1/2}}{R_{\rm s}^{1/p}R_{\rm ti}(1+z_{\rm eq})^{1/4}} \ < f < \ \dfrac{(H_{\rm r1}H_0)^{1/2}}{R_{\rm ti}(1+z_{\rm eq})^{1/4}} \end{cases} \tag{23}$$

where $\delta = 1+\sqrt{3}-1/p$. Here we can see the advantage of using the transfer function approach instead of applying the long-wavelength approximation to the full expressions for the Bogolubov β coefficient: given the initial spectrum, for a given evolution of the universe the transfer function only needs to be calculated once.

Figs. 2 and 3 show the exact final spectra and illustrate their dependence on the parameters of the model. In this model the direct detection of the stochastic background is still possible in spite of thermal inflation. As a matter of fact, because of the energy scale at which the second burst of inflation starts, the low-frequency peak falls exactly in the frequency range accessible to LISA. This is a happy coincidence and has not involved any fine tuning in the parameters of the model.

The spectra show a characteristic shape with two peaks, one for frequencies around 10^2–10^4 Hz and the other in the region 10^{-2}–10^{-4} Hz, the exact position of the peaks depending on the parameters of the model. The high-frequency peak was already discussed in Ref. [16] and it is due to the epoch of power-law inflation introduced to model the stringy phase. Since the part of the spectrum produced during the dilaton phase increases with frequency and the part of the spectrum produced in the stringy phase, which corresponds to the largest frequencies produced in this model, decreases with frequency, there must be a peak at the point where the two curves meet. Although the epoch of power-law inflation introduced to model the stringy phase is not to be taken very seriously, the fact is that during the string phase the curvature scale must be approximately constant, and therefore the part of the gravitational wave spectrum corresponding to waves produced during this period is not expected to grow with frequency. We can thus argue that this peak will most certainly be one of the key signatures in the gravitational wave spectrum produced by a PBB universe. As to the peak for frequencies around 10^{-2} Hz, it is due to the era of thermal inflation. The reason for the oscillations is the same as in the context of conventional cosmology we discussed in the previous section.

---

[1]We could consider the initial spectrum to be that at the beginning of the string epoch. However, in this case the transfer function would not be the one in Eq. (13) since it does not take into account the evolution of the gravitational waves during the stringy epoch.
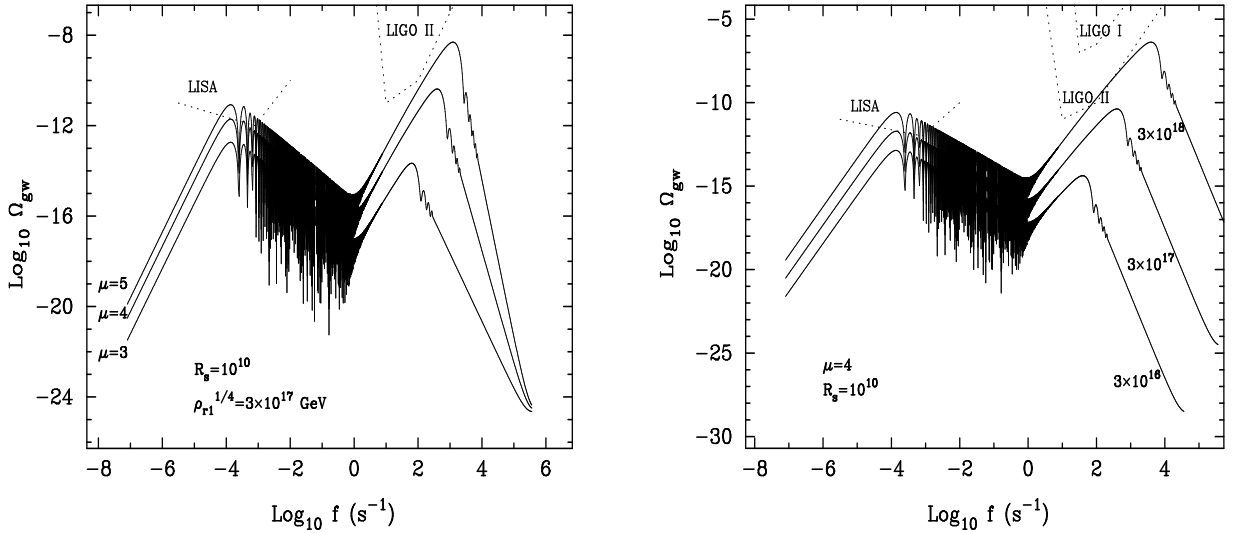
FIG. 2. The spectral energy density parameter for a model of thermal inflation in the context of PBB cosmology. The plot on the left illustrates the dependence of Ω_{gw} on the parameter μ, while the plot on the right shows the dependence on the energy scale at the end of the stringy phase. Both plots were obtained with $\rho_{\mathrm{ti}}^{1/4} = 10^8$ GeV and $R_{\mathrm{ti}} = 10^4$. We stress that the right-hand peak, near LIGO scales, violates the conditions for a perturbative calculation and is unreliable — see the text for a full discussion.

From Fig. 2, we see that larger values of μ (corresponding from Eq. (20) to larger values of p) correspond to larger values of Ω_{gw} in the high-frequency peak. This is because the slope of $\log \Omega_{\mathrm{gw}}$ is given by $2\mu - 3$.

Increasing the energy scale at the end of the string phase shifts the entire spectrum upwards (Fig. 2, right panel).

As to the dependence of Ω_{gw} on R_{s}, we see from Fig. 3 and the last line of Eq. (23) that an increase in R_{s} corresponds to a decrease in the frequency associated with the dilaton–string transition and an associated increase in the high-frequency branch of the spectrum which corresponds to the string era. Associated with this effect there is also the fact that for $R_{\mathrm{s}} > 10^{10}$ the amplitude of the low-frequency peak is larger than the high-frequency peak, since the frequency associated with the transition from thermal inflation to radiation is fixed and the branch associated with the dilaton era ($\propto f^3$) is therefore shorter.

Finally, the dependence of Ω_{gw} on R_{ti} is somewhat unexpected. When all the other parameters are fixed, larger values of R_{ti} correspond to larger Ω_{gw} in the low-frequency branch of the spectrum, contrary to what happens in all the other branches and what might have been expected, since the epoch of thermal inflation further redshifts the energy density. Once again the explanation lies in the behaviour of the frequency associated with the beginning of the low-frequency branch of the spectrum. Keeping all the other parameters fixed, this frequency decreases with $1/R_{\mathrm{ti}}$ therefore leading to a larger increase in the oscillating branch of the spectrum.

Although the high-frequency peak also falls in the frequency range accessible to LIGO, there is a technical subtlety involved in the derivation of these results which may invalidate our conclusions regarding the high-frequency peak of the spectrum, which is that this peak falls outside the domain of application of perturbative theory as we now demonstrate. Our approach to the calculation of graviton production is perturbative and therefore we must ensure that the results obtained satisfy the domain of applicability of the perturbative calculation, which in our case means that we must have $h_{ij} \ll 1$. To check the validity of this condition in this particular case we need to convert our results for the spectral energy density parameter to the dimensionless amplitude h of the perturbations.

We can obtain a rough estimate for h by noting that there is an equivalence between each of the two polarizations of the gravitational waves, $+$ and $\times$, and a scalar field φ:

$$h_{+,\times} = \sqrt{16\pi G} \, \varphi_{+,\times} \,, \tag{24}$$

from which it follows that

$$\rho_{\mathrm{gw}} \approx \frac{1}{16\pi G} \left(\frac{h'^2}{a^2} + \frac{(\nabla h)^2}{a^2} \right) \sim m_{\mathrm{Pl}}^2 \frac{h^2}{\lambda^2} \,, \tag{25}$$

where λ is a characteristic wavelength. From Eq. (25) we can now obtain the relation we require

$$h \sim \frac{\sqrt{\rho_{\mathrm{c}} \Omega_{\mathrm{gw}}}}{m_{\mathrm{Pl}}} \frac{1}{f} \,. \tag{26}$$
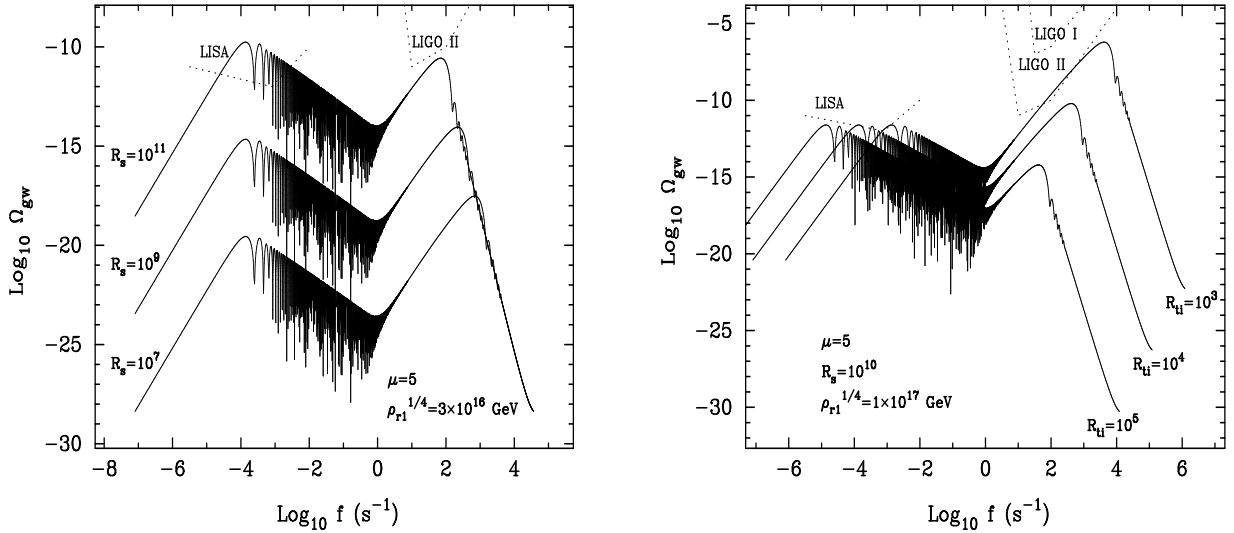
FIG. 3. The dependence of Ω_{gw} on R_{s} (left) obtained with $\rho_{\text{ti}}^{1/4} = 10^8$ GeV and $R_{\text{ti}} = 10^4$, and on R_{ti} for $\rho_{\text{ti}}^{1/4} = 10^8$ GeV and $R_{\text{s}} = 10^{10}$.

Given the values of Ω_{gw} and ρ_{c} at a given time we can estimate the corresponding values of the amplitude h. In the case under study, the problem lies in the high-frequency branch of the spectrum produced during the transition from the stringy era to radiation and therefore we must calculate Ω_{gw} at the beginning of the radiation era. We can obtain Ω_{gw} from the last line of Eq. (23), by taking also into account the redshift suffered by the frequency f during the expansion of the Universe:

$$\Omega_{\text{gw}}^{\text{rad}} = R_{\text{ti}}^4 (1 + z_{\text{eq}}) \Omega_{\text{gw}}^0 \,, \tag{27}$$

where $\Omega_{\text{gw}}^{\text{rad}}$ and Ω_{gw}^0 are the spectral energy densities of the gravitational wave background at the beginning of the radiation era and at the present. Using $R_{\text{ti}} = 10^4$, if we have $\Omega_{\text{gw}}^0 = 10^{-11}$, the threshold for the detection of the high-frequency peak by LIGO, then $\Omega_{\text{gw}}^{\text{rad}} \approx 10^{10}$. This is certainly a very large value and were we still in the regime where the perturbative calculation is valid, then this large value for $\Omega_{\text{gw}}^{\text{rad}}$ would represent an inconsistency in our model since we are assuming $\Omega_{\text{tot}} = 1$. Using $\rho_{\text{c}}^{1/4} \sim 10^{16}$ and noting that frequency is redshifted during the expansion of the Universe since the end of the stringy epoch by a factor of 10^{31}, we obtain from Eq. (26) $h_{\text{rad}} \approx 10^9$ for frequencies of the order of kHz. This means that our results for the range of frequencies where we see the first peak are not consistent with the perturbative approach and therefore they must be taken very cautiously. However, for the low-frequency peak generated due to the epoch of thermal inflation no such problem exists, and the perturbative approach employed here is justified.

This conclusion is completely general; any stochastic background generated before thermal inflation would have to be non-linearly large if it were to survive thermal inflation with an amplitude detectable by LIGO.

Notice that in this particular model, the requirement that $h \ll 1$ at the beginning of the radiation era is much stronger than the constraint $\Omega_{\text{gw}} < 10^{-5}$ imposed by nucleosynthesis [17,18]. The nucleosynthesis constraint is roughly equivalent to requiring $h \lesssim 1$ at the end of inflation for $f \sim 10^{10}$ Hz which is the high-frequency cutoff.

As a final remark, we should notice that simultaneous detection of the stochastic gravitational wave background by both LISA and LIGO in the advanced phase is not ruled out. As a matter of fact the frequencies probed by both detectors are appropriate for the detection of both peaks in the spectrum.

III. EARLY MATTER DOMINATION

Another interesting possible modification to the standard cosmology is the occurrence of an early epoch of matter domination short after inflation. A brief early epoch of matter domination may happen during the QCD phase transition [4], but we will concentrate on a more prolonged epoch, such as might be caused by temporary domination of the Universe by moduli fields [3]. The scale factor after the beginning of the first radiation epoch is given by

$$
a(\tau) = \begin{cases} a_2 \tau \,; & \tau_{\mathrm{r1}} < \tau < \tau_{\mathrm{m1}} \\[2mm] a_3 \left(\tau_3 + \tau\right)^2 \,; & \tau_{\mathrm{m1}} < \tau < \tau_{\mathrm{r2}} \\[2mm] a_4 \left(\tau_4 + \tau\right) \,; & \tau_{\mathrm{r2}} < \tau < \tau_{\mathrm{eq}} \\[2mm] a_5 \left(\tau_5 + \tau\right)^2 \,; & \tau > \tau_{\mathrm{eq}} \end{cases}
\tag{28}
$$

where once again the constants are chosen in order to ensure the continuity of a and a'. Notice that in this case the parameterization of the scale factor is slightly different from the one in the model with thermal inflation (Eq. (3)). Parametrizing the epoch of early matter domination by the energy scale at which it starts, H_{m1}, and the total expansion of the Universe during this era $R_{\mathrm{m}} \equiv a_{\mathrm{r2}}/a_{\mathrm{m1}}$, we can write all the parameters of this model in terms of H_{i}, H_{m1}, R_{m}, H_0 and z_{eq}. The value of the Hubble parameter at the time of the matter–radiation equality is in this case also given by Eq. (4) while for τ_{eq} we have

$$
\tau_{\mathrm{eq}} \approx \left(1 + z_{\mathrm{eq}}\right)^{-1/2} H_0^{-1} \,.
\tag{29}
$$

Notice that although the values of τ_{eq} given by Eq. (4) for the model of thermal inflation, and by Eq. (29) for the model with early matter domination, are the same, the first equation is exact while the second is an approximation valid for $z_{\mathrm{r2}} \gg z_{\mathrm{eq}}$, which is always the case in physically-realistic models. The difference between these two cases lies in the different parameterization for the scale factor after the beginning of the second radiation era. For the values of time at the end of inflation and the beginning and end of the early matter era we have

$$
\tau_{\mathrm{r1}} = \frac{\left(1 + z_{\mathrm{eq}}\right)^{1/4} R_{\mathrm{m}}^{1/4}}{\left(H_{\mathrm{i}} H_0\right)^{1/2}} \,, \quad \tau_{\mathrm{m1}} = \frac{\left(1 + z_{\mathrm{eq}}\right)^{1/4} R_{\mathrm{m}}^{1/4}}{\left(H_{\mathrm{m1}} H_0\right)^{1/2}} \,, \quad \tau_{\mathrm{r2}} \approx \frac{2 \left(1 + z_{\mathrm{eq}}\right)^{1/4} R_{\mathrm{m}}^{3/4}}{\left(H_{\mathrm{m1}} H_0\right)^{1/2}}
\tag{30}
$$

where the approximation in τ_{r2} is valid for $R_{\mathrm{m}} \gg 1$.

In the case of conventional inflation the scale factor in the initial epoch is given by Eq. (9) while in the PBB case the scale factor is initially given by Eq. (16).

The modifications introduced in the spectrum by the epoch of early matter domination are easy to understand in the light of what has been said before. First, the high-frequency cutoff will be redshifted. Since the usual matter era already introduces a factor $(1 + z_{\mathrm{eq}})^{1/4}$ in the high-frequency cutoff, we should also expect the extra redshift to be of the form $R_{\mathrm{m}}^{1/4}$:

$$
f_{\mathrm{r1}} = \frac{\left(H_{\mathrm{i}} H_0\right)^{1/2}}{R_{\mathrm{i1}}^{1/4} (1 + z_{\mathrm{eq}})^{1/4}} \,.
\tag{31}
$$

As before, frequencies which are outside the horizon during the early matter domination era are not affected by it and we expect the spectrum to be the same for frequencies smaller than

$$
f_{\mathrm{r2}} = a_{\mathrm{r2}} H_{\mathrm{r2}} = \frac{\left(H_{\mathrm{m}} H_0\right)^{1/2}}{R_{\mathrm{m}}^{3/4} (1 + z_{\mathrm{eq}})^{1/4}}
\tag{32}
$$

For frequencies in the interval $f_{\mathrm{m1}} < f < f_{\mathrm{r1}}$ with $f_{\mathrm{m1}} = a_{\mathrm{m1}} H_{\mathrm{m1}}$, the spectrum will be redshifted by a factor R_{m}, which corresponds to the ratio between the energy density of the gravitational waves $\rho_{gw} \sim a^{-4}$ and the energy density of matter $\rho_m \sim a^{-3}$. The slope of this part of the spectrum is not affected since these waves have frequencies larger than the frequency cutoff f_{m1} introduced by the early epoch of matter domination. For frequencies in the interval $f_{\mathrm{r2}} < f < f_{\mathrm{m1}}$ however, the early epoch of matter domination will introduce an extra factor of f^{-2} in the frequency dependence. This branch of the spectrum will smoothly join the redshifted part of the spectrum for high frequencies with the portion of the spectrum which is not affected by the early epoch of matter domination. Contrary to the previous case, once a given wavelength enters the horizon it will remain inside for the rest of its evolution, and therefore in this case there are no oscillations in the spectrum.

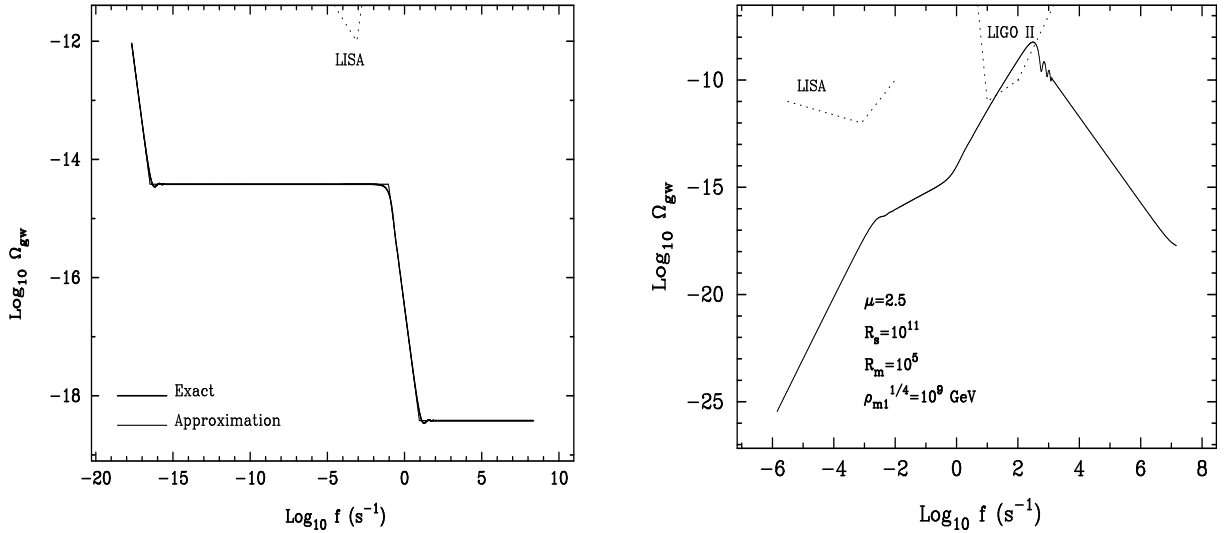Using the long-wavelength approximation we obtain

FIG. 4. The exact spectral energy density parameter for the model with an epoch of early matter domination in the context of conventional inflation (left plot) shows a suppression for $f > 10$ Hz, but in this case there are no oscillations. The suppression caused by the era of early matter domination makes the direct detection of the gravitational wave background almost impossible. The long-wavelength approximation is almost indistinguishable from the exact result except close to the transition frequencies. This plot was obtained with $\rho_i^{1/4} = 3 \times 10^{16}$ GeV, $\rho_{m1}^{1/4} = 10^{10}$ GeV and $R_m = 10^4$. The right plot is the spectral energy density parameter for a PBB model with an epoch of early matter domination. Comparing with the case of thermal inflation we see the low-frequency peak does not appear in this case. Instead, the spectrum grows as f in the intermediate region due to the early stage of matter domination. This plot was obtained with $\rho_{r1}^{1/4} = 3 \times 10^{16}$ GeV. For the PBB model, the same caveat that the perturbation calculation is unreliable applies as for the thermal inflation case.

$$
\Omega_{\mathrm{gw}} =
\begin{cases}
\dfrac{2^{3p-2}}{3^p} \dfrac{\pi^{1-p}\,\Gamma}{p^{2(1+p)}} \left(\dfrac{(\rho_c\,\rho_{r1})^{p+1}}{\rho_{\mathrm{Pl}}^{2(p+1)}\,(R_m\,(1+z_{\mathrm{eq}}))^{p-1}} \right)^{1/2} \left(\dfrac{f_{\mathrm{Pl}}}{f} \right)^{2p} & ; & H_0 < f < H_0(1+z_{\mathrm{eq}})^{1/2} \\[2em]
\dfrac{2^{3p+1}}{3^{p+1}} \dfrac{\pi^{2-p}\,\Gamma}{p^{2(1+p)}} \left(\dfrac{\rho_c^{p-1}\rho_{r1}^{p+1}}{\rho_{\mathrm{Pl}}^{2p}\,R_m^{p-1}\,(1+z_{\mathrm{eq}})^{p+1}} \right)^{1/2} \left(\dfrac{f_{\mathrm{Pl}}}{f} \right)^{2(p-1)} & ; & H_0(1+z_{\mathrm{eq}})^{1/2} < f < \dfrac{(H_{m1}H_0)^{1/2}}{R_m^{3/4}(1+z_{\mathrm{eq}})^{1/4}} \\[2em]
\dfrac{2^{3p-2}}{3^p} \dfrac{\pi^{1-p}\,\Gamma}{p^{2(1+p)}} \left(\dfrac{\rho_c^p\,\rho_{r1}^{p+1}\,\rho_{m1}}{\rho_{\mathrm{Pl}}^{2(p+1)}\,(R_m\,(1+z_{\mathrm{eq}}))^{p+2}} \right)^{1/2} \left(\dfrac{f_{\mathrm{Pl}}}{f} \right)^{2p} & ; & \dfrac{(H_{m1}H_0)^{1/2}}{R_m^{3/4}(1+z_{\mathrm{eq}})^{1/4}} < f < \dfrac{(H_{m1}H_0)^{1/2}}{(R_m(1+z_{\mathrm{eq}}))^{1/4}} \\[2em]
\dfrac{2^{3p+1}}{3^{p+1}} \dfrac{\pi^{2-p}\,\Gamma}{p^{2(1+p)}} \left(\dfrac{\rho_c^{p-1}\rho_{r1}^{p+1}}{\rho_{\mathrm{Pl}}^{2p}\,(R_m\,(1+z_{\mathrm{eq}}))^{p+1}} \right)^{1/2} \left(\dfrac{f_{\mathrm{Pl}}}{f} \right)^{2(p-1)} & ; & \dfrac{(H_{m1}H_0)^{1/2}}{(R_m(1+z_{\mathrm{eq}}))^{1/4}} < f < \dfrac{(H_{r1}H_0)^{1/2}}{(R_m(1+z_{\mathrm{eq}}))^{1/4}}
\end{cases}
$$

$$(33)$$

which confirms our expectations about the modifications introduced by the early epoch of matter domination. We also see an extra factor of R_m^{p-1} in the two low-frequency branches of the spectrum which is, as before, due to a change in the correspondence between comoving and physical scales.

Fig. 4 shows the exact result for Ω_{gw} in a model with a short period of matter domination where the Universe started with an epoch of exponential inflation $p = 1$. We see that there is a strong suppression of modes with frequencies larger than f_{mr}, the frequency corresponding to the transition from the epoch of early matter domination to the radiation era. The step structure is exactly what was found by Schwarz [4] for the QCD transition, though much more prominent as our matter-dominated era is much longer. As in the previous case, the inclusion of an epoch of early matter domination makes the direct detection of the background of gravitational waves a very difficult, if not impossible, task.

We have also analyzed a model with early matter domination in the context of PBB cosmology. Since the details of the model are not significantly different from those of the models previously analyzed we will omit them. The final result for the spectrum is shown in the Fig. 4 (right plot). In this case the low-frequency peak is absent and the early epoch of matter domination introduces a branch which behaves as f. Since the slope has the same sign as in the branch of the spectrum produced by the dilaton era, there is no peak and detection by LISA is not possible.

As in the previous case, the amplitude of the peak when extrapolated back in time will give rise to a value of $h \gg 1$ at the beginning of the radiation phase and once again the results concerning the peak must be considered very cautiously.

IV. CONCLUSIONS

The unknown cosmological behaviour before nucleosynthesis can significantly affect the present-day spectrum of cosmological gravitational waves on scales probed by interferometer experiments. In this paper we have considered two examples of this effect, one being a short late burst of inflation and the other a prolonged period of early matter domination. In each case, the amplitude of gravitational waves at high frequencies is significantly suppressed compared to the standard cosmology. In addition, the late inflation model has a very characteristic signature, in the form of a series of very strong oscillations in the spectrum.

The influence of these effects on detectability depends on the model for the initial spectrum. In conventional inflation models the situation was already extremely pessimistic anyway, so arguably a further deterioration is of little concern. In the more speculative pre big bang models of inflation, whose ability to generate observable gravitational waves has been much remarked upon, the situation is more complex; while thermal inflation makes it less likely that gravitational waves might be seen at the LIGO frequency range, it opens up a new possibility that they may be seen at the LISA range. However, the situation is quite model dependent, and there is the additional worry that on LIGO scales the usual linear theory calculations break down, so that predictions on those scales must be treated with caution. In contrast, our predictions on LISA scales satisfy the linear approximation throughout.

APPENDIX: THE BOGOLUBOV COEFFICIENTS FORMALISM

1. Exact results

The mechanism responsible for the generation of the cosmological background of gravitational waves, parametric amplification of vacuum fluctuations, was discussed by Grishchuk in the mid seventies [19]. To compute the spectrum of primordial gravitational waves we use the Bogolubov coefficient formalism [20]. We assume the transitions between the different epochs in the evolution of the Universe are instantaneous. This is in general a very good approximation and it will break down only for frequencies much larger than H at the instant of the transition. The accuracy of this approximation has been confirmed in models where the transitions are not instantaneous [21].

Since in the PBB scenario we will compute graviton production in the so-called string frame, we summarize this formalism in a form which is slightly more general than the one most commonly used [17]. The formalism laid down in this section can be applied to any theory whose effective action can be written in the form

$$S = \frac{1}{\mathcal{G}} \int \sqrt{-g}\, F(\mathcal{R}, \phi)\, d^4 x \,, \tag{A1}$$

where $\mathcal{G}$ is a constant, ϕ is a scalar field and $\mathcal{R}$ is the Ricci scalar. In general relativity $\mathcal{G} = 16\pi G$ and $F = \mathcal{R}$, while during the dilaton phase in PBB cosmology $\mathcal{G} = 1/2\lambda_\mathrm{s}^2$, where λ_s is the string-length parameter, and $F = \exp(-\phi)\mathcal{R}$ where ϕ is the dilaton. We assume the geometry of the Universe is well described by a flat Friedmann–Lemaître–Robertson–Walker (FLRW) line element

$$ds^2 = dt^2 - a^2(t)d\mathbf{x}^2 = a^2(\tau)\left(d\tau^2 - d\mathbf{x}^2\right) \,, \tag{A2}$$

where t is comoving time and τ the conformal time obtained by $d\tau = dt/a$. From now on we will always use conformal time.

To first order, the primordial gravitational waves can be represented as a transverse–traceless perturbation $h_{\mu\nu}$ of the FLRW metric [22]

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu} \,, \tag{A3}$$

where $g^{(0)}_{\mu\nu}$ is the metric of the unperturbed FLRW background. After quantization, and defining a new quantity

$$R(\tau) \equiv a(\tau) \left(\frac{\partial F}{\partial \mathcal{R}} \right)^{1/2} \,, \tag{A4}$$

the $h_{\mu\nu}$ field can be expanded as

$$h_{ij}(\mathbf{x}, \tau) = \sqrt{\mathcal{G}} \sum_{\lambda=1}^{2} \int \frac{d^3 k}{(2\pi)^{3/2} R(\tau)\sqrt{2k}} \chi_{\mathbf{k},\lambda}(\mathbf{x}, \tau) \,; \tag{A5}$$

$$\chi_{\mathbf{k},\lambda}(\mathbf{x}, \tau) = a_{\mathbf{k},\lambda} \varepsilon_{ij}^{\lambda} e^{i\mathbf{k}.\mathbf{x}} \mu_k(\tau) + a_{\mathbf{k},\lambda}^{\dagger} \varepsilon_{ij}^{\lambda\,*} e^{-i\mathbf{k}.\mathbf{x}} \mu_k^*(\tau) \,, \tag{A6}$$

where $i, j = 1, 2, 3$, the $*$ denotes the complex conjugate, $\mathbf{k}$ is the comoving wave vector, $k = |\mathbf{k}|$ and the sum in Eq. (A5) is over the two polarization states described by the polarization tensor $\varepsilon_{ij}^{\lambda}$. The $a_{\mathbf{k}}^{\dagger}$ and $a_{\mathbf{k}}$ (from now on we will drop the polarization index λ) are the creation and annihilation operators for the quanta of the field $h_{\mu\nu}$, the gravitons, and obey the canonical commutation relations

$$[a_{\mathbf{k}}, a_{\mathbf{k}'}] = \left[a_{\mathbf{k}}^{\dagger}, a_{\mathbf{k}'}^{\dagger} \right] = 0 \quad ; \quad \left[a_{\mathbf{k}}, a_{\mathbf{k}'}^{\dagger} \right] = \delta(\mathbf{k} - \mathbf{k}') \,. \tag{A7}$$

Inserting the perturbed metric into the Einstein equations, we obtain the equation which governs the behaviour of the mode functions μ_k [17,8]

$$\mu_k'' + \left(k^2 - \frac{R''}{R} \right) \mu_k = 0 \,. \tag{A8}$$

In most cases of interest, $R \propto \tau^{\alpha}$ and for such an R the solution of Eq. (A8) is given in terms of the Hankel functions $H^{(1)}$ and $H^{(2)}$. Imposing the requirement that in the limit $\tau \to -\infty$ we recover the vacuum positive frequency solution, $\mu_k \to e^{-ik\tau}$, we can write μ as

$$\mu_k = \sqrt{\frac{\pi}{2}} \sqrt{k\tau} H^{(2)}_{|\alpha - \frac{1}{2}|}(k\tau) \,. \tag{A9}$$

During its evolution, the Universe will undergo a series of transitions between different epochs with different mode functions, and therefore different vacuum states corresponding to each of these eras. The relation between the annihilation and creation operators in the initial state, $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^{\dagger}$, and in the final state, $b_{\mathbf{k}}$ and $b_{\mathbf{k}}^{\dagger}$, is given by a Bogolubov transformation [20]

$$b_{\mathbf{k}} = \alpha_k a_{\mathbf{k}} + \beta_k^* a_{-\mathbf{k}}^{\dagger} \,, \tag{A10}$$

where α_k and β_k are the Bogolubov coefficients. To obtain α_k and β_k we impose the continuity of the field h_{ij} and its first derivative at the instant of a transition between two different eras in the history of the Universe. In general relativity, where $R \equiv a$, it is always possible to make R and R' continuous, but since in the context of PBB cosmology it may not be possible to make R' continuous, as we will see later, we assume that although R is continuous R' may be discontinuous at the (instantaneous) transition between two epochs. In this case, denoting with an overbar quantities referring to the state after the transition occurred, the Bogolubov coefficients are given by [17]

$$\alpha_k = \frac{1}{2ik} \left[\mu_k \overline{\mu}_k^{*\,\prime} - \overline{\mu}_k^* \mu_k' - \mu_k \overline{\mu}_k^* \frac{\overline{R}' - R'}{R} \right] \,; \tag{A11}$$

$$\beta_k = \frac{1}{2ik} \left[\mu_k' \overline{\mu}_k - \mu_k \overline{\mu}_k' + \mu_k \overline{\mu}_k \frac{\overline{R}' - R'}{R} \right] \,, \tag{A12}$$

where all the quantities are calculated at the time of the transition. The last term in Eqs. (A11) and (A12) appears because we are not imposing the continuity of R': when R' is continuous this term is zero and we recover the usual expression for the Bogolubov coefficients.

By noting that the Wronskian for the solution Eq. (A9) of Eq. (A8) is given by

$$W = \mu_k \mu_k^{*\prime} - \mu_k^* \mu_k' = 2ik \,, \tag{A13}$$

we obtain an important relation between the Bogolubov coefficients

$$|\alpha|^2 - |\beta|^2 = 1 \,. \tag{A14}$$

The number of particles produced in mode k is given by

$$N_k = |\beta_k|^2 \,, \tag{A15}$$

and the spectral energy density parameter is defined by

$$\Omega_{\mathrm{gw}} \equiv \frac{1}{\rho_c} \frac{d\rho_{\mathrm{gw}}}{d\ln f} \,, \tag{A16}$$

where f is the frequency and ρ_c and ρ_{gw} are, respectively, the critical energy density and the energy density of gravitational waves, and is given by

$$\Omega_{\mathrm{gw}} = \frac{\hbar \omega^4}{\pi^2 c^3 \rho_c} N_k \,, \tag{A17}$$

where the angular frequency $\omega = 2\pi f = ck/a$.

When we consider two or more transitions in the history of the Universe, the final Bogolubov coefficients are given by the composition of the Bogolubov coefficients in each transition, according to

$$\alpha_k = \alpha_{1k} \alpha_{2k} + \beta_{1k} \beta_{2k}^* \,; \tag{A18}$$
$$\beta_k = \beta_{1k} \alpha_{2k}^* + \alpha_{1k} \beta_{2k} \,. \tag{A19}$$

From now on we shall drop the index k in α and β. Since at any instant modes with wavelengths larger than the horizon have not yet gone through a complete oscillation, they will not contribute to the energy density of gravitational waves at that given time and we are therefore forced to introduce a low-frequency cutoff for frequencies smaller than

$$\omega_{\mathrm{lf}}(\tau) = 2\pi H(\tau) \,. \tag{A20}$$

As we are assuming that all the transitions are instantaneous, a high-frequency cutoff must also be introduced. In an instantaneous transition modes with arbitrarily high frequency will be excited. However, a real transition will take a finite amount of time, and therefore these high-frequency modes are unphysical, since modes with frequencies much larger than the rate of expansion of the Universe will suffer many oscillations during the transition and will not be affected by it. The value of the high-frequency cutoff is given by [23,24]

$$\omega_{\mathrm{hf}}(\tau) = 2\pi \frac{a_{\mathrm{tr}} H_{\mathrm{tr}}}{a(\tau)} \,. \tag{A21}$$

where the subscript 'tr' refers to the time when the transition takes place. This rough argument for the introduction of the high-frequency cutoff can be formalized in a more rigorous way via Parker's adiabatic theorem [25], which states that modes with frequencies larger than Eq. (A21) will be exponentially suppressed. Using a modification of the formalism presented in this section which allows for continuous transitions, and introducing a toy model for the transitions, it is also possible to see that modes with frequencies larger than the cutoff frequency given in Eq. (A21) are indeed exponentially suppressed [21].

2. Long-wavelength approximation and transfer function

Although the application of the formalism just described is straightforward, for models involving several transitions the final result for Ω_{gw} tends to become very long and not very instructive, and an approximation is due at this point. The high-frequency cutoff given by Eq. (A21) can be written in comoving coordinates as

$$k_{\mathrm{hf}} = 2\pi a_{\mathrm{tr}} H_{\mathrm{tr}} \,. \tag{A22}$$

This corresponds to frequencies which are just entering the horizon at a given transition. Since $k < k_{\mathrm{hf}}$ and $k_{\mathrm{hf}} \tau \approx (a'/a)\tau \approx 1$, then $k\tau < 1$ in most cases of interest and we can replace the Hankel functions appearing in the mode

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functions (Eq. (A9)) by their leading terms. This approximation is usually very good for most purposes as we can see in Fig. 1. Since modes satisfying $k\tau < 1$ have wavelengths larger than the Hubble radius, this approximation is known as the long-wavelength approximation.

The long-wavelength approximation can be used to generate a transfer function for the gravitational wave spectrum. Once we have an initial spectrum then, for a given subsequent evolution of the Universe, the same transfer function can always be applied to the initial spectrum, no matter what its form or origin is, to produce the spectrum at the present.

Noting that the leading term in the expansion of the Hankel function $H^{(2)}$ is given by

$$H_\nu^{(2)}(k\tau) = \frac{i\,2^\nu \csc(\nu\pi)}{\Gamma(1-\nu)} \frac{1}{(k\tau)^\nu} + \cdots , \tag{A23}$$

from Eq. (A9) we can immediately see that in the long wavelength approximation the mode functions are given by

$$\mu \propto (k\tau)^{-\nu+1/2} . \tag{A24}$$

After a succession of n transitions we should expect the final β to be given by a product of terms of the form $(k\tau_1)^{\nu_1} \cdots (k\tau_n)^{\nu_n}$ with the possibility that some of the terms may repeat themselves in the product. After we replace the mode functions by their long-wavelength approximation, from Eqs. (A11), (A12), (A18), (A19) we get

$$\beta \approx (k\tau_1)^{-\nu_1} \left(\frac{\tau_1}{\tau_2}\right)^{-\nu_2} \cdots \left(\frac{\tau_{n-1}}{\tau_n}\right)^{-\nu_n} (k\tau_n)^{-\nu_{n+1}} , \quad k_{n+1} < k < k_n$$

where we dropped all the numerical factors. The index $n+1$ appears in the last term since the nth transition links the epochs n and $n+1$. The number of particles for the branch of the spectrum corresponding to the nth transition will therefore be given by

$$N_k = |\beta|^2 \approx \tau_1^{-2(\nu_1+\nu_2)}\tau_2^{-2(\nu_3-\nu_2)} \cdots \tau_{n-1}^{-2(\nu_n-\nu_{n-1})}\tau_n^{-2(\nu_{n+1}-\nu_n)} k^{-2(\nu_{n+1}+\nu_1)} , \quad k_{n+1} < k < k_n . \tag{A25}$$

From Eqs. (A11), (A12), (A18), (A19) we see that the final Bogolubov coefficients will be a sum of products of Hankel functions, each term with $2n$ of these, in the case of the nth transition. We can therefore ask ourselves why the final dependence on k is just $k^{-2(\nu_{n+1}+\nu_1)}$ and does not involve any of the exponents $\nu_2 \cdots \nu_{n-1}$ which should be contributed by the previous transitions. The reason for this cancellation can be understood by noticing that

$$H_\nu^{(2)}(x) = J_\nu(x) - i\,Y_\nu(x)$$

where J and Y are the Bessel functions of 1st and 2nd kind. The Y function contributes the leading term $x^{-\nu}$ while to lower order $J \sim x^\nu$. The intermediate powers of k cancel because $J \sim x^\nu$ and in the end we are left with $\beta \propto k^{-(\nu_1+\nu_{n+1})}$.

We can use Eq. (A25) to build a transfer function for the evolution of the gravitational waves. The initial spectrum (the spectrum after the first transition) takes the form

$$N_k^{(0)} = \tau_1^{-2(\nu_1+\nu_2)}k^{-2(\nu_1+\nu_2)} . \tag{A26}$$

We are looking for a transfer function, which we shall call $G(k)$, such that $N_k = G(k)N_k^{(0)}$. From Eq. (A25) we can easily see that the required $G(k)$ is given by:

$$G(k) = \begin{cases} \tau_2^{-2(\nu_3-\nu_2)} \cdots \tau_{n-1}^{-2(\nu_n-\nu_{n-1})}\tau_n^{-2(\nu_{n+1}-\nu_n)}k^{-2(\nu_{n+1}-\nu_2)} & k_{n+1} < k < k_n \\ \tau_2^{-2(\nu_3-\nu_2)} \cdots \tau_{n-1}^{-2(\nu_n-\nu_{n-1})}k^{-2(\nu_n-\nu_2)} & k_n < k < k_{n-1} \\ \cdots & \cdots \\ \tau_2^{-2(\nu_3-\nu_2)}k^{-2(\nu_3-\nu_2)} & k_3 < k < k_2 \\ 1 & k_2 < k < k_1 \end{cases} \tag{A27}$$

Note that $G(k)$ is independent of the initial spectrum and can therefore, once calculated for a given cosmology, be applied to a variety of different choices of initial spectrum. Usually, due to requirements of continuity of the scale factor, the time dependence of a during the nth epoch is given in terms of $\tau_n' + \tau$ where τ_n' is a constant. As a consequence the argument of the Hankel functions will be $k(\tau_n' + \tau)$ and therefore, in Eq. (A27) τ_n must be replaced by $\tau_n' + \tau_n$.

In our model of thermal inflation there is however a further complication: the cutoffs associated with the transitions from the first epoch of radiation to the epoch of thermal inflation, $k_{\rm rti}$, and from thermal inflation to the final radiation epoch, $k_{\rm tir}$, satisfy $k_{\rm rti} < k_{\rm tir}$, contrary to the common situation where later transition will give rise to cutoffs at smaller frequencies. Assuming this situation involves transitions between branches i and $i+1$ and between $i+1$ and $i+2$ $(i > 1)$, the transfer function will take the form

$$G(k) = \begin{cases} \cdots & \cdots \\ \tau_2^{-2(\nu_3-\nu_2)}\cdots\tau_{i-1}^{-2(\nu_i-\nu_{i-1})}\tau_i^{-2(\nu_{i+1}-\nu_i)}\tau_{i+1}^{-2(\nu_{i+2}-\nu_{i+1})}k^{-2(\nu_{i+2}-\nu_2)} & k_{i+2} < k < k_i \\ \tau_2^{-2(\nu_3-\nu_2)}\cdots\tau_{i-1}^{-2(\nu_i-\nu_{i-1})}\tau_{i+1}^{-2(\nu_{i+1}+\nu_{i+2})}k^{-2(\nu_i+\nu_{i+1}+\nu_{i+2}-\nu_2)} & k_i < k < k_{i+1} \\ \tau_2^{-2(\nu_3-\nu_2)}\cdots\tau_{i-1}^{-2(\nu_i-\nu_{i-1})}k^{-2(\nu_i-\nu_2)} & k_{i+1} < k < k_{i-1} \\ \cdots & \cdots \end{cases} \tag{A28}$$

Obviously, since here we are taking only the lowest-order terms, with this approximation we cannot recover the oscillations seen in Figs. 1, 2 and 3, but at least we can get the correct dependence on frequency.

[1] K. Thorne, gr-qc/9704042; B. Allen and J. D. Romano, Phys. Rev. D **59**, 102001 (1999).
[2] D. H. Lyth and E. D. Stewart, Phys. Rev. Lett. **75**, 201 (1995); D. H. Lyth and E. D. Stewart, Phys. Rev. D **53**, 784 (1996).
[3] A. M. Green, A. R. Liddle and A. Riotto, Phys. Rev. D **56**, 7559 (1997).
[4] D. J. Schwarz, Mod. Phys. Lett. **A13**, 2771 (1998).
[5] M. Gasperini and G. Veneziano, Astropart. Phys. **1**, 317 (1993).
[6] M. Gasperini and G. Veneziano, Mod. Phys. Lett. **A8**, 3701 (1993).
[7] M. Gasperini and M. Giovannini, Phys. Lett. B **282**, 36 (1992); R. Brustein, M. Gasperini, M. Giovannini, and G. Veneziano, Phys. Lett. B **361**, 45 (1995); R. Brustein, M. Gasperini, M. Giovannini, V. F. Mukhanov, and G. Veneziano, Phys. Rev. D **51**, 6744 (1995).
[8] M. Gasperini and M. Giovannini, Phys. Rev. D **47**, 1519 (1993).
[9] B. Allen and R. Brustein, Phys. Rev. D **55**, 3260 (1997).
[10] D. Polarski and A. A. Starobinsky, Nucl. Phys. B **385**, 623 (1992).
[11] F. Lucchin and S. Matarrese, Phys. Rev. D **32**, 1316 (1985).
[12] A. R. Liddle, Phys. Rev. D **49**, 3805 (1994); (E) **51**, 4603 (1995).
[13] G. Veneziano, Phys. Lett. B **265**, 287 (1991);
[14] G. Veneziano, in *Proc. of the 4th Course of the Int. School of Astrophysics D. Chalonge* (Erice, 1995), edited by N. Sanchez and A. Zichichi (Kluwer Academic, Dordrecht, 1995), p. 285, hep-th/9512091; M. Gasperini, in *Proc. of the 4th Course of the Int. School of Astrophysics D. Chalonge* (Erice, 1995), edited by N. Sanchez and A. Zichichi (Kluwer Academic, Dordrecht, 1995), p. 305, hep-th/9509127.
[15] M. Gasperini and G. Veneziano, Phys. Rev. D **50**, 2519 (1994).
[16] M. Galluccio, M. Litterio and F. Occhionero, Phys. Rev. Lett. **79**, 970 (1997).
[17] M. R. de Garcia Maia, *The Stochastic Background of Gravitational Waves*, DPhil. Thesis (1994) University of Sussex; M. R. de Garcia Maia and J. D. Barrow, Phys. Rev. D **50**, 6262 (1994).
[18] B. Allen, in *Relativistic Gravitation and Gravitational Radiation*, edited by J.-A. Marck and J.-P. Lasota (Cambridge University Press, 1997).
[19] L. P. Grishchuk, Sov. Phys. JETP **40**, 409 (1975); L. P. Grishchuk, Ann. NY Acad. Sci. **302**, 439 (1977).
[20] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space, Cambridge Monographs on Mathematical Physics* (Cambridge University Press, Cambridge, 1992). For applications to the case of graviton production see L. F. Abbott and M. B. Wise, Nucl. Phys. **B244**, 541 (1984) and Ref. [23].
[21] R. G. Moorhouse, A. B. Henriques, and L. E. Mendes, Phys. Rev. D **50**, 2600 (1994); L. E. Mendes, A. Henriques, and R. G. Moorhouse, Phys. Rev. D **52**, 2083 (1995), S. Koranda and B. Allen, Phys. Rev. D **52**, 1902 (1995).
[22] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (W. H. Freeman, New York, 1973).
[23] B. Allen, Phys. Rev. D **37**, 2078 (1988).
[24] M. R. de Garcia Maia, Phys. Rev. D **48**, 647 (1993).
[25] L. Parker, Phys. Rev. **183**, 1057 (1969).